Session 7: solutions

Exercise 1:

The scaling hypothesis tells

$$f(t,h) = b^{-d} f(t b^{y_t}, h b^{y_h})$$

then

- $$m = -\frac{\partial f}{\partial h}\Big|_{h=0} = b^{-d+y_h} f_2'(t b^{y_t}, 0)$$

f_2' = deriv. w.r.t. the 2nd variable

choosing $b = |t|^{-1/y_t}$ we have

$$m = |t|^{\frac{d-y_h}{y_t}} f\left(\frac{t}{|t|}, 0\right) = |t|^{\frac{d-y_h}{y_t}} f_2'(\pm 1, 0)$$

$\pm$ depends if $t \gtrless 0$

then

$$\boxed{\beta = \frac{d-y_h}{y_t}}$$

- $$m = -\frac{\partial f}{\partial h}\Big|_{t=0} = b^{-d+y_h} f(0, h b^{y_h})$$

choosing $b = |h|^{-1/y_h}$ we have

$$m = |h|^{\frac{d-y_h}{y_h}} f_2'(0, \pm 1)$$

$$\boxed{\frac{1}{\delta} = \frac{d-y_h}{y_h}}$$

- $$\chi = \frac{dm}{dh}\bigg|_{h=0} = -\frac{\partial^2 f}{\partial h^2}\bigg|_{h=0} = b^{-d+2y_h} f_2''(t\,b^{y_t}, h\,b^{y_h})\Big|_{h=0} =$$
$$= b^{-d+2y_h} f_2''(t\,b^{y_t}, 0)$$

choosing

$$b = |t|^{-1/y_t} \quad \text{we get}$$

$$\chi = |t|^{\frac{d-2y_h}{y_t}} f_2''(\pm 1, 0)$$

$$\gamma = \frac{2y_h - d}{y_t}$$

- $$C \propto \frac{\partial^2 f}{\partial T^2}\bigg|_{h=0} \propto \frac{\partial^2 f}{\partial t^2}\bigg|_{h=0} =$$
$$= b^{-d+2y_t} f_1''(t\,b^{y_t}, 0)$$

choosing $b = |t|^{-1/y_t}$ we have

$$C \propto |t|^{\frac{d-2y_t}{y_t}} f_1''(\pm 1, 0)$$

$$\alpha = \frac{2y_t - d}{y_t}$$

- Rushbrooke inequality:

$$\alpha + 2\beta + \gamma \geq 2$$

$$\frac{2y_t - d}{y_t} + 2\frac{d - y_h}{y_t} + \frac{2y_h - d}{y_t} = \frac{2y_t - d + 2d - 2y_h + 2y_h - d}{y_t} =$$

$$= \frac{2y_t}{y_t} = 2 \quad \checkmark$$

- Griffiths inequality:

$$\alpha + \beta(1+\delta) \geq 2$$

$$\frac{2y_t - d}{y_t} + \frac{d - y_h}{y_t}\left(1 + \frac{y_h}{d - y_h}\right) =$$

$$= \frac{2y_t - d}{y_t} + \frac{d - y_h}{y_t}\,\frac{d}{d - y_h} = \frac{2y_t - d + d}{y_t} = 2 \quad \checkmark$$

- Fisher's inequality

$$\gamma \geq (2-\eta)\nu$$

here we need η:

$E = -J\sum_{\langle ij\rangle} s_i s_j - h\sum_i s_i$

$\downarrow$

Let's write the susceptibility again

$$m = -\frac{1}{N}\frac{\partial F}{\partial h} = -\frac{\beta}{N}\frac{1}{Z}\sum_{\{s_i\}}\left(\sum_i s_i\right)e^{-\beta E(\{s_i\})}$$

then

$$\chi = -\frac{\beta}{N}\left\{+\frac{\beta}{Z^2}\left(\sum_{\{s_i\}}\left(\sum_i s_i\right)e^{-\beta E(\{s_i\})}\right)^2 + \right.$$

$$\left. -\frac{\beta}{Z}\sum_{\{s_i\}}\left(\sum_i s_i\right)^2 e^{-\beta E(\{s_i\})}\right\} =$$

$$= \frac{\beta^2}{N}\left\{\sum_{\{s_i\}}\sum_i\sum_j (s_i s_j)\frac{e^{-\beta E}}{Z} - \left[\sum_{\{s_i\}}\left(\sum_i s_i\right)\frac{e^{-\beta E}}{Z}\right]^2\right\} =$$

$$= \frac{\beta^2}{N}\sum_j c(\vec{r}_j) \qquad \vec{r}_j = \vec{r}_i - \vec{r}_j$$

$\downarrow$ go to continuum limit

$$\chi = \frac{\beta^2}{N}\int d\vec{r}\, c(\vec{r}) \simeq \frac{\beta^2}{N}\int d\vec{r}\,\frac{1}{r^{d-2+\eta}}e^{-r/\xi}$$

$\uparrow$ goes to a finite value if $r \to 0$

then

$$\chi = \frac{\beta^2}{N}\left[\int \frac{d\vec{r}}{\xi^d} \frac{\xi^{d-2+\eta}}{r^{d-2+\eta}} e^{-r/\xi}\right] \xi^d \xi^{-(d-2+\eta)} =$$

$$= \frac{\beta^2}{N} \xi^{2-\eta} \cdot \text{const} \propto |t|^{-\nu(2-\eta)}$$

$$\left[\nu = \frac{1}{y_t}\right] \qquad \gamma = (2-\eta)\nu$$

- Josephson's relation:

$$2-\alpha \geq d\nu \quad \Rightarrow \quad 2-\alpha-d\nu \geq 0$$

$$2 - \frac{2y_t - d}{y_t} - \frac{d}{y_t} = \frac{2y_t - 2y_t + d - d}{y_t} = 0$$